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Integral Representations for Solutions of Some Types of the Beltrami Equations

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Abstract—We obtain integral representations for solutions of some types of the Beltrami equations. These representations allow us to prove analogs of some classical complex analysis for these solutions.

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Introduction. We consider the Beltrami equation $\overline{\partial}\phi = \mu\partial\phi$, $|\mu(z)| < 1$. As usual, here

$$\overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

and $\mu(z)$ is a given Lebesgue measurable function. This equation is one of important generalizations of the Cauchy–Riemann system. It has a great body of applications (see, e.g., [1–3]).

Let us fix domain Δ, non-constant analytic function $f(z)$ defined there, and a positive constant α. Put $g(z) := f(z)|f(z)|^{2\alpha}$. Immediate substitution shows that this function satisfies the Beltrami equation

$$\overline{\partial}\phi = \beta\frac{f}{\overline{f}}\frac{\overline{f}'}{f'}\partial\phi, \tag{1}$$

where $\beta := \frac{\alpha}{1+\alpha}$. In what follows we consider just this Beltrami equation.

In the following Item we obtain an integral representation for its solutions, which is a generalization of the Cauchy integral. In the last Item we apply it for the proof of certain analogs of some classical results of complex analysis for solutions of the Beltrami equation.

1. Integral representation. Denote for brevity $h = f/f'$ and $\overline{\partial}^{\beta} = \overline{\partial} - \beta\frac{h}{\overline{h}}\partial$.

Lemma. *Let D be a finite domain with piecewise-smooth boundary Γ, and its closure be lying in Δ. If functions ϕ and ψ are continuous in $\overline{D}$, have there integrable with degree larger two partial derivatives, and $\overline{\partial}^{\beta}\phi = 0$ in D, then*

$$\iint_D \frac{\phi(\zeta)}{h(\zeta)}\overline{\partial}^{\beta}\psi(\zeta)dx_\zeta\, dy_\zeta = \frac{1}{2i}\int_\Gamma \frac{\phi(t)\psi(t)}{h(t)}dt + \frac{\beta}{2i}\int_\Gamma \frac{\phi(t)\psi(t)}{\overline{h(t)}}d\overline{t} - S, \tag{2}$$

$$S = \pi(1-\beta)\sum_{j=1}^{m}\phi(z_j)\psi(z_j),$$

where $z_1, z_2, \dots, z_m$ stand for zeros of function f in the domain D. If f has not zeros in this domain, then $S = 0$.

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